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The vibrations of a thin elastic orthotropic circular cylindrical shell with free and hinged edges☆

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ABSTRACT

The problem of the existence of natural oscillations of a thin elastic orthotropic circular closed cylindrical shell with free and hinge-mounted ends and of an open cylindrical shell with free and hinge-mounted edges, when the two boundary generatrices are hinge-mounted is investigated. Dispersion equations and asymptotic formulae for finding the natural frequencies of possible vibration modes are obtained using the system of equations corresponding to the classical theory of orthotropic cylindrical shells. A mechanism is proposed by means of which the vibrations can be separated into possible types. Approximate values of the dimensionless characteristic of the natural frequency and the attenuation characteristic of the corresponding vibration modes are obtained using the examples of closed and open orthotropic cylindrical shells of different lengths.

It is well known that plane and flexural waves exist, independently of one another, at the free edge of a semi-infinite orthotropic plate.[1–3] When the plate is bent, two of these types of motion are coupled, giving initially two new types of waves, localized at the edge (mainly tangential and mainly flexural). Transformation of one type of wave motion into the other occurs at the free end of a thin cylindrical elastic shell. In this transformation of the waves, complex distribution patterns of the frequencies of natural oscillations of finite and infinite cylindrical shells with a free edge occur, depending on the geometrical and mechanical parameters of the shell.[4–19]

Using dispersion equations and asymptotic formulae for these dispersion equations, obtained below, by varying the geometry of the shells and the mechanical properties of the material one can control the spectrum by shifting either the origin of the spectrum or the points of condensation from the undesirable resonance region.[20,21]

1. Fundamental equations and formulation of the boundary-value problems

It is assumed that the generatrices of the cylindrical shell are orthogonal to the shell edges. Curvilinear coordinates (α, β), where $\alpha(0 \le \alpha \le l)$ and $\beta(0 \le \beta \le s)$ are the length of the generatrices and the length of the arc of the directing circle (see the figure) respectively, are introduced on the middle surface of the shell. The equations of the vibrations of a shell, which correspond to the classical theory of orthotropic cylindrical shells are used, and are written in the chosen curvilinear coordinates α, β:

$$-B_{11}\frac{\partial^2 u_1}{\partial\alpha^2} - B_{66}\frac{\partial^2 u_1}{\partial\beta^2} - (B_{12}+B_{66})\frac{\partial^2 u_2}{\partial\alpha\partial\beta} + \frac{B_{12}}{R}\frac{\partial u_3}{\partial\alpha} = \lambda u_1$$

$$-(B_{12}+B_{66})\frac{\partial^2 u_1}{\partial\alpha\partial\beta} - B_{66}\frac{\partial^2 u_2}{\partial\alpha^2} - B_{22}\frac{\partial^2 u_2}{\partial\beta^2} + \frac{B_{22}}{R}\frac{\partial u_3}{\partial\beta} -$$

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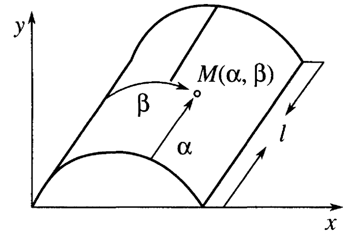

$$-\frac{\mu^4}{R^2}\left(4B_{66}\frac{\partial^2 u_2}{\partial\alpha^2}+B_{22}\frac{\partial^2 u_2}{\partial\beta^2}\right)-\frac{\mu^4}{R}\left(B_{22}\frac{\partial^3 u_3}{\partial\beta^3}+(B_{12}+4B_{66})\frac{\partial^3 u_3}{\partial\beta\partial\alpha^2}\right)=\lambda u_2$$

$$\mu^4\left(B_{11}\frac{\partial^4 u_3}{\partial\alpha^4}+2(B_{12}+2B_{66})\frac{\partial^4 u_3}{\partial\alpha^2\partial\beta^2}+B_{22}\frac{\partial^4 u_3}{\partial\beta^4}\right)+$$

$$+\frac{\mu^4}{R}\left(B_{22}\frac{\partial^3 u_2}{\partial\beta^3}+(B_{12}+4B_{66})\frac{\partial^3 u_2}{\partial\beta\partial\alpha^2}\right)-\frac{B_{12}}{R}\frac{\partial u_1}{\partial\alpha}-\frac{B_{22}}{R}\frac{\partial u_2}{\partial\beta}+\frac{B_{22}}{R^2}u_3=\lambda u_3 \tag{1.1}$$

Here u_1, u_2 and u_3 are the projections of the displacement vector in the directions α, β and normal to the shell surface, respectively, R is the radius of the directing circle of the middle surface, $\mu^4 = h^2/12$ (h is the shell thickness), $\lambda = \omega^2\rho$, where ω is the angular frequency of natural vibrations, ρ is the density of the material and B_{ij} are the coefficients of elasticity.[22]

We will consider the following boundary conditions

$$\left.\frac{\partial u_1}{\partial\alpha}+\frac{B_{12}}{B_{11}}\left(\frac{\partial u_2}{\partial\beta}-\frac{u_3}{R}\right)\right|_{\alpha=0}=\left.\frac{\partial u_2}{\partial\alpha}+\frac{\partial u_1}{\partial\beta}+\frac{4\mu^4}{R}\left(\frac{\partial^2 u_3}{\partial\alpha\partial\beta}+\frac{1}{R}\frac{\partial u_2}{\partial\alpha}\right)\right|_{\alpha=0}=0$$

$$\left.\frac{\partial^2 u_3}{\partial\alpha^2}+\frac{B_{12}}{B_{11}}\left(\frac{\partial^2 u_3}{\partial\beta^2}+\frac{1}{R}\frac{\partial u_2}{\partial\beta}\right)\right|_{\alpha=0}=\left.\frac{\partial^3 u_3}{\partial\alpha^3}+\frac{B_{12}+4B_{66}}{B_{11}}\left(\frac{\partial^3 u_3}{\partial\alpha\partial\beta^2}+\frac{1}{R}\frac{\partial^2 u_2}{\partial\alpha\partial\beta}\right)\right|_{\alpha=0}=0 \tag{1.2}$$

$$\left.\frac{\partial u_1}{\partial\alpha}+\frac{B_{12}}{B_{11}}\left(\frac{\partial u_2}{\partial\beta}-\frac{u_3}{R}\right)\right|_{\alpha=l}=u_2|_{\alpha=l}=u_3|_{\alpha=l}=\left.\frac{\partial^2 u_3}{\partial\alpha^2}+\frac{B_{12}}{B_{11}}\left(\frac{\partial^2 u_3}{\partial\beta^2}+\frac{1}{R}\frac{\partial u_2}{\partial\beta}\right)\right|_{\alpha=l}=0 \tag{1.3}$$

$$u_i(\alpha,\beta)=u_i(\alpha,\beta+s),\quad i=1,2,3 \tag{1.4}$$

$$\left.\frac{B_{12}}{B_{22}}\frac{\partial u_1}{\partial\alpha}+\frac{\partial u_2}{\partial\beta}-\frac{u_3}{R}\right|_{\beta=0,s}=u_1|_{\beta=0,s}=u_3|_{\beta=0,s}=\left.\frac{B_{12}}{B_{22}}\frac{\partial^2 u_3}{\partial\alpha^2}+\frac{\partial^2 u_3}{\partial\beta^2}+\frac{1}{R}\frac{\partial u_2}{\partial\beta}\right|_{\beta=0,s}=0 \tag{1.5}$$

Boundary conditions (1.2)–(1.4) correspond to the closed cylindrical shell; relations (1.2) express the free-edge conditions when α = 0, Eqs (1.3) are the conditions of the hinge-mounted edge when α = l, and Eq.(1.4) are the vibration periodicity conditions, where s is the total length of the directing circle of the middle surface. Boundary conditions (1.2), (1.3) and (1.5) correspond to a cylindrical shell of open profile; relations (1.5) are the conditions for hinge mounting along the directrices β = 0 and β = s, where s is the length of the arc of the circle of the middle surface between the hinge-mounted generatrices (see the figure).

It can be proved that problems (1.1)–(1.4) and (1.1)–(1.3), (1.5) are self-conjugate and have a non-negative discrete spectrum with a limit point at +∞ ([12], p.362).

2. Derivation and analysis of the characteristic equations

In the first, second and third equations of system (1.1) the spectral parameter λ is formally replaced by λ_1, λ_2 and λ_3 respectively. For subsequent calculations it is more convenient to reduce the system of equations (1.1) (with the changes indicated) to the following system of equations

$$\left(\Gamma+\frac{\mu^4}{R^2}DG\right)u_1=\frac{1}{R}\frac{\partial}{\partial\alpha}\left(Au_3+\mu^4\frac{B_{22}(B_{12}+B_{66})}{B_{11}B_{66}}\frac{\partial^2}{\partial\beta^2}Lu_3+\frac{\mu^4}{R^2}\frac{B_{12}}{B_{11}}\frac{B_{22}}{B_{66}}Du_3\right)$$

$$\left(\Gamma + \frac{\mu^4}{R^2}DG\right)u_2 = \frac{1}{R}\frac{\partial}{\partial\beta}(Bu_3 - \mu^4 LGu_3)$$

$$\Gamma\Omega u_3 + \frac{1}{R^2}\left\{\left(\Gamma - B\frac{\partial^2}{\partial\beta^2} - \frac{B_{12}}{B_{22}}A\frac{\partial^2}{\partial\alpha^2}\right)u_3 + \mu^4\left(DG\Omega + 2LB\frac{\partial^2}{\partial\beta^2}\right)u_3 + \right.$$

$$\left. + \frac{\mu^4}{R^2}\left(DB + \frac{B_{12}}{B_{11}}D\frac{\partial^2}{\partial\alpha^2}\right)u_3 - \mu^8 L^2 G\frac{\partial^2 u_3}{\partial\beta^2}\right\} = 0 \quad (2.1)$$

where

$$A = \frac{B_{12}}{B_{11}}\frac{\partial^2}{\partial\alpha^2} - \frac{B_{22}}{B_{11}}\frac{\partial^2}{\partial\beta^2} + \frac{B_{12}}{B_{11}}\frac{\lambda_2}{B_{66}},\ B = B_1\frac{\partial^2}{\partial\alpha^2} + \frac{B_{22}}{B_{11}}\frac{\partial^2}{\partial\beta^2} + \frac{B_{22}}{B_{11}}\frac{\lambda_1}{B_{66}},\ B_1 = \frac{B_{11}B_{22} - B_{12}^2 - B_{12}B_{66}}{B_{11}B_{66}}$$

$$\Gamma = \frac{\partial^4}{\partial\alpha^4} + B_2\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + \frac{B_{22}}{B_{11}}\frac{\partial^4}{\partial\beta^4} + \left(\frac{\lambda_1}{B_{11}} + \frac{\lambda_2}{B_{66}}\right)\frac{\partial^2}{\partial\alpha^2} + \left(\frac{\lambda_1 B_{22}}{B_{11}B_{66}} + \frac{\lambda_2}{B_{11}}\right)\frac{\partial^2}{\partial\beta^2} + \frac{\lambda_1\lambda_2}{B_{11}B_{66}}$$

$$B_2 = \frac{B_{11}B_{22} - B_{12}^2 - 2B_{12}B_{66}}{B_{11}B_{66}},\quad D = \frac{4B_{66}}{B_{22}}\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2},\quad G = \frac{B_{22}}{B_{66}}\frac{\partial^2}{\partial\alpha^2} + \frac{B_{22}}{B_{11}}\frac{\partial^2}{\partial\beta^2} + \frac{B_{22}}{B_{11}}\frac{\lambda_1}{B_{66}}$$

$$L = \frac{\partial^2}{\partial\beta^2} + \frac{B_{12} + 4B_{66}}{B_{22}}\frac{\partial^2}{\partial\alpha^2},\quad \Omega = \mu^4\left(\frac{B_{11}}{B_{22}}\frac{\partial^4}{\partial\alpha^4} + \frac{2(B_{12} + 2B_{66})}{B_{22}}\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + \frac{\partial^4}{\partial\beta^4}\right) - \frac{\lambda_3}{B_{22}} \quad (2.2)$$

We introduce the following notation: $k = 2\pi n_0/s$, $n_0 \in N$ for a closed cylindrical shell and $k = \pi/s$ for a cylindrical shell of open profile. Suppose $R^{-1} = kr_0/2$, where r_0 is a dimensionless parameter. We will seek a solution of system (1.1) in the form

$$(u_1, u_2, u_3) = (u_{sm}\sin km\beta,\ \upsilon_{cm}\cos km\beta,\ \sin km\beta)\exp(k\chi\alpha),\quad m = 1, 2, \ldots \quad (2.3)$$

Here m is the wave number, u_{sm} and υ_{sm} are undetermined coefficients, and χ is the undetermined attenuation constant. Conditions (1.4) and (1.5) are then automatically satisfied, and the problems are solved in a similar way if the parameter k is given different values. Substituting expressions (2.3) into system (2.1) we obtain

$$\left(c_m + \frac{r_0^2}{4}a^2 g_m d_m\right)u_{sm} = \frac{r_0\chi}{2}\left\{a_m - a^2 m^2\frac{B_{22}(B_{12} + B_{66})}{B_{11}B_{66}}l_m + \frac{r_0^2}{4}a^2\frac{B_{22}B_{12}}{B_{11}B_{66}}d_m\right\} \quad (2.4)$$

$$\left(c_m + \frac{r_0^2}{4}a^2 g_m d_m\right)\upsilon_{cm} = \frac{r_0 m}{2}\{b_m - a^2 g_m l_m\} \quad (2.5)$$

$$R_{mm}c_m + \frac{r_0^2}{4}\left\{c_m + m^2 b_m - \frac{B_{12}}{B_{22}}\chi^2 a_m + a^2(R_{mm}g_m d_m - 2m^2 l_m b_m) + \right.$$

$$\left. + \frac{r_0^2}{4}a^2 d_m\left(b_m + \frac{B_{12}}{B_{11}}\chi^2\right) + a^4 m^2 g_m l_m^2\right\} = 0 \quad (2.6)$$

$$a_m = \frac{B_{12}}{B_{11}}\chi^2 + \frac{B_{22}}{B_{11}}m^2 + \frac{B_{12}}{B_{11}}\eta_2^2,\quad b_m = B_1\chi^2 - \frac{B_{22}}{B_{11}}m^2 + \frac{B_{22}}{B_{11}}\eta_1^2,\quad a^2 = \mu^4 k^2$$

$$c_m = \chi^4 + \frac{B_{22}}{B_{11}}m^4 - B_2\chi^2 m^2 + \left(\frac{B_{66}}{B_{11}}\eta_1^2 + \eta_2^2\right)\chi^2 - \left(\frac{B_{22}}{B_{11}}\eta_1^2 + \frac{B_{66}}{B_{11}}\eta_2^2\right)m^2 + \frac{B_{66}}{B_{11}}\eta_1^2\eta_2^2$$

$$d_m = \frac{4B_{66}}{B_{11}}\chi^2 - m^2,\quad g_m = \frac{B_{22}}{B_{66}}\chi^2 - \frac{B_{22}}{B_{11}}m^2 + \frac{B_{22}}{B_{11}}\eta_1^2,\quad l_m = \frac{B_{12} + 4B_{66}}{B_{22}}\chi^2 - m^2$$

$$R_{mm} = a^2\left(\frac{B_{11}}{B_{22}}\chi^4 - \frac{2(B_{12} + 2B_{66})m^2}{B_{22}}\chi^2 + m^4\right) - \frac{B_{66}}{B_{22}}\eta_3^2;\quad \eta_i^2 = \frac{\lambda_i}{B_{66}k^2},\quad i = 1, 2, 3 \quad (2.7)$$

Suppose χ_j (j = 1,2,3,4) are pairwise different zeros of Eq.(2.6) with non-positive real parts. Then $\chi_5 = -\chi_1$, $\chi_6 = -\chi_2$, $\chi_7 = -\chi_3$, $\chi_8 = -\chi_4$ are also pairwise different zeros of this equation. Suppose $(u_1^{(j)}, u_2^{(j)}, u_3^{(j)})$ are non-trivial solutions of system (1.1) of the form (2.3) with $\chi = \chi_j$ (j = 1,2,. . .,8) respectively. Representing the solution of problems (1.1)–(1.4) and (1.1)–(1.3), (1.5) in the form

$$u_i = \sum_{j=1}^{8} w_j u_i^{(j)}, \quad i = 1, 2, 3$$

and taking boundary conditions (1.2) and (1.3) into account, we obtain the following system of equations

$$\sum_{j=1}^{8} \frac{M_{ij}^{(m)} w_j}{c_m^{(j)} + \frac{r_0^2}{4} a^2 g_m^{(j)} d_m^{(j)}} = 0, \quad i = 1, 2, \dots, 8 \tag{2.8}$$

Here

$$M_{1j}^{(m)} = \chi_j^2 a_m^{(j)} - \frac{B_{12}}{B_{11}} m^2 b_m^{(j)} - \frac{B_{12}}{B_{11}} c_m^{(j)} + \frac{r_0^2}{4} a^2 \frac{B_{12} B_{22}}{B_{11}^2} d_m^{(j)} (m^2 - \eta_1^2) -$$
$$- a^2 m^2 \frac{B_{22}}{B_{11}} l_m^{(j)} \left(\chi_j^2 + \frac{B_{12}}{B_{11}} m^2 - \frac{B_{12}}{B_{11}} \eta_1^2 \right)$$

$$M_{2j}^{(m)} = m\chi_j \left\{ a_m^{(j)} + b_m^{(j)} + a^2 \left[4c_m^{(j)} - l_m^{(j)} \left(\frac{B_{22}}{B_{66}} \chi_j^2 + \frac{B_{12} B_{22}}{B_{11} B_{66}} m^2 + \frac{B_{22}}{B_{11}} \eta_1^2 \right) \right] + \right.$$
$$\left. + \frac{r_0^2}{4} a^2 \left(4b_m^{(j)} + \frac{B_{12} B_{22}}{B_{11} B_{66}} d_m^{(j)} - 4a^2 \frac{B_{12}}{B_{22}} \chi_j^2 g_m^{(j)} \right) \right\}$$

$$M_{3j}^{(m)} = \left(\chi_j^2 - \frac{B_{12}}{B_{11}} m^2 \right) c_m^{(j)} + \frac{r_0^2}{4} \left[a^2 \chi_j^2 g_m^{(j)} \left(\frac{4B_{66}}{B_{22}} \chi_j^2 - \frac{B_{11} B_{22} - B_{12}^2}{B_{11} B_{22}} m^2 \right) - \frac{B_{12}}{B_{11}} m^2 b_m^{(j)} \right]$$

$$M_{4j}^{(m)} = \chi_j \left\{ \left(\chi_j^2 - \frac{B_{12} + 4B_{66}}{B_{11}} m^2 \right) c_m^{(j)} + \frac{r_0^2}{4} \left[a^2 \chi_j^2 g_m^{(j)} \left(\chi_j^2 - \frac{B_{12} + 4B_{66}}{B_{11}} m^2 \right) - \right. \right.$$
$$\left. \left. - \frac{B_{12} + 4B_{66}}{B_{11}} m^2 b_m^{(j)} + a^2 m^2 \frac{B_{12} + 4B_{66}}{B_{11}} g_m^{(j)} l_m^{(j)} \right] \right\} \tag{2.9}$$

$$M_{5j}^{(m)} = M_{1j}^{(m)} \exp(z_j), \quad M_{6j}^{(m)} = (b_m^{(j)} - a^2 g_m^{(j)} l_m^{(j)}) \exp(z_j), \quad z_j = k\chi_j l$$

$$M_{7j}^{(m)} = \left(c_m^{(j)} + \frac{r_0^2}{4} a^2 g_m^{(j)} d_m^{(j)} \right) \exp(z_j), \quad M_{8j}^{(m)} = M_{3j}^{(m)} \exp(z_j), \quad j = 1, 2, \dots, 8$$

The superscript j in the brackets denotes that the corresponding function is taken when $\chi = \chi_j$.
In order that system (2.8) should have a non-trivial solution, it is necessary and sufficient that

$$\det \left\| M_{ij}^{(m)} \right\|_{i, j = 1}^{8} = 0 \tag{2.10}$$

Numerical analysis shows that the left-hand side of this equality becomes small when any two roots of Eq. (2.6) become close to one another. This considerably complicates the calculations and can lead to the appearance of false solutions. It turns out that the factor on the left-hand side of Eq. (2.10), which approaches zero as the roots approach one another, can be separated out.

To do this we introduce the following notation

$$x_j = \frac{\chi_j}{m}, \quad j = 1, 2, \ldots, 8; \quad \eta_{im} = \frac{\eta_i}{m}, \quad i = 1, 2, 3; \; \varepsilon_m = \frac{r_0}{2m}; \quad \delta_m = 1 + 4a^2 m^2 \varepsilon_m^2$$

$$[z_i z_j] = \frac{\exp(z_i) - \exp(z_j)}{z_i - z_j} kml, \quad [z_i z_j z_k] = \frac{[z_i z_j] - [z_i z_k]}{z_j - z_k} kml$$

$$[z_1 z_2 z_3 z_4] = \frac{[z_1 z_2 z_3] - [z_1 z_2 z_4]}{z_3 - z_4} kml$$

$$\sigma_1 = \sigma_1(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + x_4$$

$$\sigma_2 = \sigma_2(x_1, x_2, x_3, x_4) = x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4$$

$$\sigma_3 = \sigma_3(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4$$

$$\sigma_4 = \sigma_4(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4, \quad \bar{\sigma}_k = \sigma_k(x_1, x_2, x_3, 0), \quad \bar{\bar{\sigma}}_k = \sigma_k(x_1, x_2, 0, 0)$$
$$k = 1, 2, 3, 4 \tag{2.11}$$

Here $\bar{\sigma}_4 = \bar{\bar{\sigma}}_4 = \bar{\bar{\sigma}}_3 = 0$.

Suppose f_n (n = 1,2,. . .,6) is a symmetrical polynomial of the n-th degree of the variables x_1, x_2, x_3 and x_4. It is well known that it can be uniquely expressed in terms of elementary symmetrical polynomials. Putting

$$f_n = f_n(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \quad \bar{f}_n = f_n(\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3, 0), \quad f_n = f_n(\sigma_1, \sigma_2, 0, 0)$$
$$n = 1, 2, \ldots, 6 \tag{2.12}$$

$$f_1 = \sigma_1, \; f_2 = \sigma_1^2 - \sigma_2, \; f_3 = \sigma_1^3 - 2\sigma_1\sigma_2 + \sigma_3, \; f_4 = \sigma_1^4 - 3\sigma_1^2\sigma_2 + \sigma_2^2 + 2\sigma_1\sigma_3 - \sigma_4$$
$$\bar{f}_5 = \bar{\sigma}_1^5 - 4\bar{\sigma}_1^3\bar{\sigma}_2 + 3\bar{\sigma}_1\bar{\sigma}_2^2 + 3\bar{\sigma}_1^2\bar{\sigma}_3 - 2\bar{\sigma}_2\bar{\sigma}_3, \quad \bar{\bar{f}}_6 = \bar{\bar{\sigma}}_1^6 - 5\bar{\bar{\sigma}}_1^4\bar{\bar{\sigma}}_2 + 6\bar{\bar{\sigma}}_1^2\bar{\bar{\sigma}}_2^2 - \bar{\bar{\sigma}}_2^3 \tag{2.13}$$

and performing elementary actions on the columns of determinant (2.10), we obtain

$$\det\left\|M_{ij}^{(m)}\right\|_{i,j=1}^{8} = m^{37}\mathbf{K}^2 \exp(-z_1 - z_2 - z_3 - z_4) \det\left\|m_{ij}\right\|_{i,j=1}^{8}$$
$$\mathbf{K} = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) \tag{2.14}$$

$$m_{11} = Hx_1^4 + d_1x_1^2 + d_2, \quad m_{12} = H\bar{\bar{f}}_3 + d_1\bar{\bar{f}}_1, \quad m_{13} = H\bar{f}_2 + d_1, \quad m_{14} = Hf_1$$

$$m_{21} = Tx_1^5 + d_3x_1^3 + d_4x_1, \; m_{22} = T\bar{\bar{f}}_4 + d_3\bar{\bar{f}}_2 + d_4, \; m_{23} = T\bar{f}_3 + d_3\bar{f}_1, \; m_{24} = Tf_2 + d_3$$

$$m_{31} = \delta_m x_1^6 + d_5x_1^4 + d_6x_1^2 + d_7, \quad m_{32} = \delta_m\bar{\bar{f}}_5 + d_5\bar{\bar{f}}_3 + d_6\bar{\bar{f}}_1, \quad m_{33} = \delta_m\bar{f}_4 + d_5\bar{f}_2 + d_6$$

$$m_{34} = \delta_m f_3 + d_5f_1; \; m_{41} = \delta_m x_1^7 + d_8x_1^5 + d_9x_1^3 + d_{10}x_1, \quad m_{42} = \delta_m\bar{\bar{f}}_6 + d_8\bar{\bar{f}}_4 + d_9\bar{\bar{f}}_2 + d_{10}$$

$$m_{43} = \delta_m\bar{f}_5 + d_8\bar{f}_3 + d_9\bar{f}_1, \quad m_{44} = \delta_m f_4 + d_8f_2 + d_9$$

$$m_{i5} = (-1)^{i+1}m_{i1}\exp(z_1), \quad m_{i6} = (-1)^{i+1}(m_{i2}\exp(z_2) + m_{i1}[z_1z_2])$$

$$m_{i7} = (-1)^{i+1}(m_{i3}\exp(z_3) + m_{i2}[z_2z_3] + m_{i1}[z_1z_2z_3])$$

$$m_{i8} = (-1)^{i+1}(m_{i4}\exp(z_4) + m_{i3}[z_3z_4] + m_{i2}[z_2z_3z_4] + m_{i1}[z_1z_2z_3z_4])$$

$$m_{5i} = m_{1i+4}, \quad m_{5i+4} = m_{1i}; \quad i = 1, 2, 3, 4$$

$$n_{61} = Fx_1^4 + \theta_1x_1^2 + \theta_2, \quad n_{62} = F\bar{\bar{f}}_3 + \theta_1\bar{\bar{f}}_1, \quad n_{63} = F\bar{f}_2 + \theta_1, \quad n_{64} = Ff_1$$

$$n_{71} = \delta_m x_1^4 + \theta_3x_1^2 + \theta_4, \quad n_{72} = \delta_m\bar{\bar{f}}_3 + \theta_3\bar{\bar{f}}_1, \quad n_{73} = \delta_m\bar{f}_2 + \theta_3, \quad n_{74} = \delta_m f_1$$

$$m_{i1} = n_{i1}\exp(z_1), \; m_{i2} = n_{i2}\exp(z_2) + n_{i1}[z_1z_2], \; m_{i3} = n_{i3}\exp(z_3) + n_{i2}[z_2z_3] + n_{i1}[z_1z_2z_3]$$

$$m_{i4} = n_{i4}\exp(z_4) + n_{i3}[z_3z_4] + n_{i2}[z_2z_3z_4] + n_{i1}[z_1z_2z_3z_4]$$

$$m_{ij+4} = n_{ij}; \quad i = 6, 7, \quad j = 1, 2, 3, 4; \quad m_{8i} = m_{3i+4}, \quad m_{8i+4} = m_{3i}; \quad i = 1, 2, 3, 4$$

$$H = -a^2m^2\frac{B_{12} + 4B_{66}}{B_{11}}, \quad T = -a^2m^2\delta_m\frac{B_{12}}{B_{66}}, \quad F = -a^2m^2\frac{B_{12} + 4B_{66}}{B_{66}}$$

$$d_1 = \frac{B_{11}B_{22} - B_{12}^2}{B_{11}^2} - \frac{B_{12}B_{66}}{B_{11}^2}\eta_{1m}^2 + 4a^2m^2\varepsilon_m^2\frac{B_{12}B_{66}}{B_{11}^2}(1 - \eta_{1m}^2) +$$

$$+\frac{a^2m^2}{B_{11}^2}(B_{11}B_{22} - B_{12}^2 - 4B_{12}B_{66} + B_{12}(B_{12} + 4B_{66})\eta_{1m}^2)$$

$$d_2 = \frac{B_{12}}{B_{22}}(1 - \eta_m^2)(B_{66}\eta_{2m}^2 + a^2m^2B_{22}(1 - \varepsilon_m^2))$$

$$d_3 = a^2m^2\left(\frac{B_{12}}{B_{11}} - 3B_1 + 4\eta_{2m}^2 - \frac{B_{12}}{B_{11}}\eta_{1m}^2\right) + \frac{B_{11}B_{22} - B_{12}^2}{B_{11}B_{66}}\delta_m + 4a^4m^4\varepsilon_m^2\frac{B_{12}}{B_{11}}(1 - \eta_{1m}^2)$$

$$d_4 = \frac{B_{22}}{B_{11}}\eta_{1m}^2 + \frac{B_{12}}{B_{11}}\eta_{2m}^2 + a^2m^2\left(\frac{B_{22}(B_{12} + 4B_{66})}{B_{11}B_{66}} - 3\frac{B_{22}}{B_{11}}\eta_{1m}^2 - 4\frac{B_{66}}{B_{11}}\eta_{2m}^2(1 - \eta_{1m}^2)\right) -$$

$$-a^2m^2\varepsilon_m^2\frac{B_{22}}{B_{11}}\left(\frac{(B_{12} + 4B_{66})}{B_{66}} - 4\eta_{1m}^2\right) \qquad (2.15)$$

$$d_5 = \frac{B_{66}}{B_{11}}\eta_{1m}^2 + \eta_{2m}^2 - B_1 - a^2m^2\varepsilon_m^2\left(B_1 + \frac{B_{12}+4B_{66}}{B_{11}} - \frac{4B_{66}}{B_{11}}\eta_{1m}^2\right)$$

$$d_6 = \frac{B_{12}}{B_{11}}B_2 + \frac{B_{22}}{B_{11}} - \left(\frac{B_{11}B_{22}+B_{12}B_{66}}{B_{11}^2}\eta_{1m}^2 + \frac{B_{12}+B_{66}}{B_{11}}\eta_{2m}^2\right) + \frac{B_{66}}{B_{11}}\eta_{1m}^2\eta_{2m}^2 +$$

$$+\varepsilon_m^2\left(a^2m^2\frac{B_{22}(B_{11}B_{22}-B_{12}^2)}{B_{11}^2B_{22}}(1-\eta_{1m}^2) - \frac{B_{12}}{B_{11}}B_1\right)$$

$$d_7 = \frac{B_{12}}{B_{11}}(1-\eta_{1m}^2)\left(\frac{B_{22}}{B_{11}}\varepsilon_m^2 - \frac{B_{22}}{B_{11}} + \frac{B_{66}}{B_{11}}\eta_{2m}^2\right)$$

$$d_8 = \frac{B_{66}}{B_{11}}\eta_{1m}^2 + \eta_{2m}^2 - B_1 - \frac{4B_{66}}{B_{11}} - a^2m^2\varepsilon_m^2\left(B_2 + \frac{4B_{66}-2B_{12}}{B_{11}} - \frac{4B_{66}}{B_{11}}\eta_{1m}^2\right)$$

$$d_9 = \frac{B_{66}}{B_{11}}\eta_{1m}^2\eta_{2m}^2 - \frac{B_{11}B_{22}+B_{12}B_{66}+4B_{66}^2}{B_{11}^2}\eta_{1m}^2 - \frac{B_{12}+5B_{66}}{B_{11}}\eta_{2m}^2 + \frac{B_{22}}{B_{11}} +$$

$$+\frac{B_{12}+4B_{66}}{B_{11}}B_2 + \varepsilon_m^2\left(a^2m^2\frac{B_{11}B_{22}-B_{12}^2-4B_{12}B_{66}}{B_{11}^2}(1-\eta_{1m}^2) - \frac{(B_{12}+4B_{66})}{B_{11}}B_1\right)$$

$$d_{10} = \frac{B_{12}+4B_{66}}{B_{11}}(1-\eta_{1m}^2)\left(\frac{B_{22}}{B_{11}}\varepsilon_m^2 - \frac{B_{22}}{B_{11}} + \frac{B_{66}}{B_{11}}\eta_{2m}^2\right)$$

$$\theta_1 = B_1 - a^2m^2\frac{B_{12}+4B_{66}}{B_{11}}\eta_{1m}^2 + a^2m^2\frac{B_{22}B_{11}+B_{12}B_{66}+4B_{66}^2}{B_{11}B_{66}},\ \theta_2 = -\frac{B_{22}}{B_{11}}(1+a^2m^2)(1-\eta_{1m}^2)$$

$$\theta_3 = \frac{B_{66}}{B_{11}}\eta_{1m}^2 + \eta_{2m}^2 - B_2 - \varepsilon_m^2a^2m^2\left(\frac{B_{22}}{B_{66}} + \frac{4B_{66}}{B_{11}}(1-\eta_{1m}^2)\right),$$

$$\theta_4 = (1-\eta_{1m}^2)\left(\frac{B_{22}}{B_{11}} - \frac{B_{66}}{B_{11}}\eta_{2m}^2 + \varepsilon_m^2a^2m^2\frac{B_{22}}{B_{11}}\right)$$

Eq. (2.10) is equivalent to the equation

$$\det\|m_{ij}\|_{i,j=1}^{8} = 0 \tag{2.16}$$

Taking into account possible relations between λ_1, λ_2 and λ_3, we conclude that Eq. (2.16) defines the frequencies of the corresponding modes of vibration.

When $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, Eq. (2.6) is the characteristic equation of system (1.1); when $k = 2\pi n_0/s$, $n_0 \in N$, $m \in N$ Eq. (2.16) is the dispersion equation of (1.1)–(1.4), and when $k = \pi/s$, $m \in N$ it is the dispersion equation of problem (1.1)–(1.3), (1.5).

3. The asymptotic of dispersion equation (2.16) when $r_0 \to 0$

When using the previous formulae we will assume that $\eta_{1m} = \eta_{2m} = \eta_{3m} = \eta_m = \eta/m$. Then, when $r_0 \to 0$ Eq.(2.6) is converted into the set of equations

$$c_m = \chi^4 + \frac{B_{22}}{B_{11}}m^4 - B_2\chi^2m^2 + \frac{B_{11}+B_{66}}{B_{11}}\eta^2\chi^2 - \frac{B_{22}+B_{66}}{B_{11}}\eta^2m^2 + \frac{B_{66}}{B_{11}}\eta^4 = 0 \tag{3.1}$$

$$R_{mm} = a^2\left(\frac{B_{11}}{B_{22}}\chi^4 - \frac{2(B_{12}+2B_{66})m^2}{B_{22}}\chi^2 + m^4\right) - \frac{B_{66}}{B_{22}}\eta^2 = 0 \tag{3.2}$$

which are characteristic equations for the equations of plane and flexural vibrations of a plate, respectively.[6,10,16] The roots χ/m of Eqs (3.1) and (3.2) with non-positive real parts are denoted y_1, y_2 and y_3, y_4 respectively.

It was proved in Ref. 6 that when

$$\varepsilon_m \ll 1,\quad y_i \neq y_j,\quad i \neq j \tag{3.3}$$

the roots $(\chi/m)^2$ of Eq. (2.6) can be represented in the form

$$x_i^2 = y_i^2 + \alpha_i^{(m)}\varepsilon_m^2 + \beta_i^{(m)}\varepsilon_m^4 + \ldots, \quad i = 1, 2, 3, 4 \tag{3.4}$$

Under conditions (3.3), taking relations (2.9) and (3.4) into account and also the fact that

$$\frac{M_{3i}}{m^6} = \frac{M_{4i}}{m^7} = \frac{M_{7i}}{m^4} = \frac{M_{8i}}{m^5} = O(\varepsilon_m^2), \quad i = 1, 2 \tag{3.5}$$

we can reduce Eq. (2.16) to the form

$$\det\|m_{ij}\|_{i,j=1}^{8} = (1-\eta_m^2)N(\eta_m)N_0(\eta_m)\mathbf{K}_3^2(\eta_m)\bar{G}(\eta_m)\bar{L}(\eta_m) + O(\varepsilon_m^2) = 0$$

$$N(\eta_m) = (y_1+y_3)(y_2+y_3)(y_1+y_4)(y_2+y_4), \quad N_0(\eta_m) = N(\eta_m)(y_1+y_2)(y_3+y_4)$$

$$\bar{G}(\eta_m) = \mathbf{K}_1(\eta_m)(1-\exp(2(z_3+z_4))) + (y_3+y_4)\mathbf{K}_5(\eta_m)(\exp(z_3)+\exp(z_4))[z_3z_4]$$

$$\bar{L}(\eta_m) = K_2(\eta_m)(1-\exp(2(z_1+z_2))) + (y_1+y_2)\mathbf{K}_4(\eta_m)(\exp(z_1)+\exp(z_2))[z_1z_2] \tag{3.6}$$

$$\mathbf{K}_{1,5}(\eta_m) = y_3^2y_4^2 \pm 4\frac{B_{66}}{B_{11}}y_3y_4 - \left(\frac{B_{12}}{B_{11}}\right)^2, \quad \mathbf{K}_{4,2}(\eta_m) = (1-\eta_m^2)\left(\frac{B_{11}B_{22}-B_{12}^2}{B_{11}B_{66}} - \eta_m^2\right) \pm \eta_m^2 y_1y_2$$

$$\mathbf{K}_3(\eta_m) = N_1(\eta_m) + a^2m^2N_2(\eta_m) + a^4m^4N_3(\eta_m)$$

$$N_1(\eta_m) = \left(\frac{B_{12}B_{66}}{B_{11}^2}B_1 - \frac{B_{12}B_{66}B_{22}}{B_{11}^3}\right)\eta_m^2 + \frac{B_{22}(B_{11}B_{22}-B_{12}^2)}{B_{11}^3}$$

$$N_2(\eta_m) = -\frac{2B_{22}(B_{11}B_{22}-B_{12}^2)}{B_{11}^3} - \frac{1}{B_{66}B_{11}^3}(B_{11}B_{22}B_{12}^2 - B_{12}^4 + 2B_{11}B_{22}B_{12}B_{66} -$$

$$-6B_{12}^3B_{66} - 10B_{12}^2B_{66}^2 - 2B_{22}B_{66}B_{12}^2 - 8B_{12}B_{22}B_{66}^2 - 8B_{12}B_{66}^3 - 4B_{11}B_{22}B_{66}^2 -$$

$$-4B_{22}B_{66}^2)\eta_m^2 + \frac{(B_{12}+4B_{66})(B_{11}-B_{66})B_{12}}{B_{11}^3}\eta_m^4$$

$$N_3(\eta_m) = \frac{B_{22}^2(B_{12}+B_{66})}{B_{11}^2B_{66}}\left\{\frac{(B_{12}+4B_{66})^2}{B_{11}B_{22}} - \frac{(B_{12}+4B_{66})}{B_{22}}B_2 + 1 -\right.$$

$$\left.-\frac{(B_{12}+4B_{66})(B_{12}B_{22}+B_{12}B_{66}+3B_{22}B_{66}+4B_{66}^2-B_{11}B_{22})}{B_{11}B_{22}^2}\eta_m^2 + \frac{(B_{12}+4B_{66})^2B_{66}}{B_{11}B_{22}^2}\eta_m^4\right\} \tag{3.7}$$

It follows from Eq. (3.6) that when $\varepsilon_m \to 0$ Eq. (2.16) splits into the equations

$$\bar{L}(\eta_m) = 0, \quad \bar{G}(\eta_m) = 0, \quad \mathbf{K}_3(\eta_m) = 0 \tag{3.8}$$

Of these, the first two are dispersion equations of plane and flexural vibrations in the analogous problem of an orthotropic plate-strip with free and hinge-mounted edges ($k=2\pi_0/s$) or a rectangular plate with free and three hinge-mounted edges ($k=\pi/s$). Plane vibrations of a cylindrical shell correspond to the roots of the third equation; it manifests itself as a result of using the equation of the corresponding classical theory of orthotropic cylindrical shells. When $\eta_m > 1$ the plane vibrations become non-decaying vibrations.

If y_1, y_2 and y_3, y_4 are the roots of Eq. (3.1) and (3.2) with negative real parts respectively, then when $ml \to \infty$ Eq. (2.16) becomes the equation

$$\det\|m_{ij}\|_{i,j=1}^{8} = (1-\eta_m^2)N(\eta_m)N_0(\eta_m)\mathbf{K}_1(\eta_m)\mathbf{K}_2(\eta_m)\mathbf{K}_3^2(\eta_m) +$$

$$+O(\varepsilon_m^2) + \sum_{j=1}^{4} O(\exp(z_j)) = 0 \tag{3.9}$$

from which it follows that when $\varepsilon_m \to$ and $ml \to \infty$, dispersion Eq. (2.16) splits into the following equations

$$\mathbf{K}_1(\eta_m) = 0, \quad \mathbf{K}_2(\eta_m) = 0, \quad \mathbf{K}_3(\eta_m) = 0 \tag{3.10}$$

Of these the first two are dispersion equations of flexural and plane vibrations of a semi-infinite orthotropic plate with free edge ($k = 2\pi n_0/s$) or of a semi-infinite orthotropic plate-strip with free end when there is a hinge-mounting on the side edges ($k = \pi/s$) respectively.[6,9,16] Consequently, for small ε_m and large ml, the roots of Eq. (3.10) are approximate values of the roots of Eq. (3.9).

4. The asymptotic of dispersion equation (2.16) when l→ ∞

When using the previous formulae we will assume that $\chi_1, \chi_2, \chi_3, \chi_4$, (the roots of Eq.(2.6)) have negative real parts. Then, Eq. (2.16) can be reduced to the form

$$\det\|m_{ij}\|_{i,j=1}^{8} = \det\|m_{ij}\|_{i,j=1}^{4}\cdot\det\|m_{ij}\|_{i,j=5}^{8} + \sum_{j=1}^{4} O(\exp(k\chi_j l)) = 0 \qquad (4.1)$$

whence it follows that when $l \to \infty$, Eq. (2.16) splits into the equations

$$\det\|m_{ij}\|_{i,j=1}^{4} = 0, \quad \det\|m_{ij}\|_{i,j=5}^{8} = 0 \qquad (4.2)$$

The first of these, when $m \in N$, defines all kinds of localized natural vibrations of the free end of a semi-infinite orthotropic circular closed cylindrical shell ($k = 2\pi n_0/s$) or a cylindrical shell of open profile ($k = \pi/s$) when there is a hinge mounting on the boundary generatrices.[6,9,16] If $\varepsilon_m \to 0$, we have

$$\det\|m_{ij}\|_{i,j=1}^{4} = N(\eta_m)\{\mathbf{K}_1(\eta_m)\mathbf{K}_2(\eta_m)\mathbf{K}_3(\eta_m) + O(\varepsilon_m^2)\} \qquad (4.3)$$

$$\det\|m_{ij}\|_{i,j=5}^{8} = N_0(\eta_m)\{(1-\eta_m^2)\mathbf{K}_3(\eta_m) + O(\varepsilon_m^2)\} \qquad (4.4)$$

Consequently, taking formulae (4.1), (4.3) and (4.4) into account, we conclude that dispersion Eq. (2.16) takes the form (3.9). In Table 1 we show values of some roots (η_m) of Eq. (3.8) for a perspex plate with the following mechanical parameters[17]

$$\rho = 2.4\cdot 10^3\ \text{Kg/M}^3, \quad E_1 = 6.37\cdot 10^{10}\ \text{N/M}^2, \quad E_2 = 1.47\cdot 10^{10}, \quad G = 4.9\cdot 10^9,$$
$$\nu_1 = 0.26, \quad \nu_2 = 0.06 \qquad (4.5)$$

where $h = 1/50$, $k = \pi/4$, 1 and 0.7851 and l = 15.5. A numerical analysis shows that at the free edge of the plate-strip, when the elastic edge is hinge-mounted, and at the free edge of a rectangular plate, when the remaining three edges are hinge-mounted, localized vibrations may appear. When ml → 0 the frequencies of the localized vibrations at the free edge of a plate-strip and of a rectangular plate approach the frequencies of a semi-infinite plate and the frequencies of a semi-infinite plate-strip respectively (see Table 1 and also Table 1 in Ref. 6).

Table 1

m	$\bar{L}(\eta_m)=0, k=\pi/4$		$\bar{L}(\eta_m)=0, k=1$			$\bar{G}(\eta_m)=0, k=\pi/4$		$\bar{G}(\eta_m)=0, k=1$	
	l=15	l=5	l=15	l=5	m	l=15	l=5	l=15	l=15
1	0.9652	0.9085	0.9708	0.9272	1				
2	0.9777	0.9515	0.9798	0.9603	2				
3	0.9808	0.9652	0.9816	0.9708	3				
4	0.9817	0.9717	0.9820	0.9757	4			0.0403	
5	0.9820	0.9754	0.9821	0.9783	5	0.0397		0.0504	
6	0.9821	0.9770	0.9821	0.9798	6	0.0475		0.0605	
7	0.9821	0.9792	0.9821	0.9808	7	0.0554		0.0705	
8	0.9821	0.9802	0.9821	0.9813	8	0.0633		0.0806	
9	0.9821	0.9808	0.9821	0.9816	9	0.0712		0.0907	
10	0.9821	0.9813	0.9821	0.9818	10	0.0791		0.1008	
11	0.9821	0.9815	0.9821	0.9820	11	0.0871		0.1108	
12	0.9821	0.9817	0.9821	0.9820	12	0.0950		0.1209	0.1209
13	0.9821	0.9819	0.9821	0.9821	13	0.1029		0.1310	0.1310
14	0.9821	0.9820	0.9821	0.9821	14	0.1108		0.1411	0.1411
15	0.9821	0.9820	0.9821	0.9821	15	0.1187	0.1187	0.1512	0.1512
16	0.9821	0.9821	0.9821	0.9821	16	0.1266	0.1266	0.1612	0.1612
m	$K_3(\eta_m)=0, k=0.7851$		$K_3(\eta_m)=0, k=1$		17	0.1345	0.1346	0.1713	0.1713
93			0.1896		18	0.1424	0.1425	0.1814	0.1814
94			0.2848		19	0.1504	0.1504	0.1914	0.1914
95			0.3533		20	0.1583	0.1583	0.2015	0.2015
100			0.5694		90	0.7122	0.7122	0.9068	0.9068
105			0.7042		95	0.7518	0.7518	0.9572	0.9572
110			0.8031		100	0.7914	0.7914	1.0076	1.0076
115			0.8804		110	0.8705	0.8705	1.1084	1.1084
120	0.3008		0.9431		120	0.9496	0.9496	1.2091	1.2091
125	0.5035		0.9952		125	0.9892	0.9892	1.2595	1.2595

Table 2

l	*m*	$\eta_1=\eta_2=\eta_3=\eta$		$\eta_1=\eta_2=0,\ \eta_3=\eta$		$\eta_1=\eta_2=\eta,\ \eta_3=0$		$\eta_2=\eta_3=\eta,\ \eta_1=0$	
		$k\chi_0/m$	η/m	$k\chi_0/m$	η/m	$k\chi_0/m$	η/m	$k\chi_0/m$	η/m
	2					−0.0739	0.8372e		
	4					−0.0472	0.9791e		
	5	−0.0049	0.0396b	−0.0049	0.0396b	−0.0448	0.9810e	−0.0049	0.0396b
	6	−0.0098	0.0475b	−0.0095	0.0475b	−0.0440	0.9816e	−0.0098	0.0475b
	7	−0.0116	0.0554b	−0.0116	0.0554b	−0.0436	0.9817e	−0.0116	0.0554b
	8	−0.0125	0.0634b	−0.0125	0.0633b	−0.0435	0.9820e	−0.0125	0.0633b
	9	−0.0130	0.0712b	−0.0130	0.0712b	−0.0434	0.9820e	−0.0130	0.0712b
15	10	−0.00133	0.0791b	−0.0133	0.0791b	−0.0434	0.9821e	−0.133	0.0791b
	100	−0.0138	0.7911b	−0.0138	0.7911b	−0.0433	0.9821e	−0.0138	0.7911b
	110	−0.0138	0.8702b	−0.0138	0.8702b	−0.0433	0.9821e	−0.0138	0.8702b
	120	−0.2153	0.3008n			−0.2153	0.3008n	−0.2258	0.3001t
		−0.0138	0.9493b	−0.0138	0.9493b	−0.0433	0.9821e	−0.0138	0.9493b
	125	−0.1956	0.5035n			−0.1956	0.5035n	−0.2264	0.5021t
		−0.0433	0.9821e			−0.0433	0.9821e		
		−0.0138	0.9888b	−0.0138	0.9888b			−0.0138	0.9888b
	5					−0.1113	0.8122e		
	15					−0.0434	0.9820e		
	16	−0.0074	0.1266b	−0.0074	0.1266b	−0.0434	0.9820e	−0.0074	0.1266b
	17	−0.0088	0.1345b	−0.0088	0.1345b	−0.0433	0.9821e	−0.0088	0.1345b
	18	−0.0099	0.1424b	−0.0099	0.1424b	−0.0433	0.9821e	−0.0099	0.1424b
	19	−0.0106	0.1503b	−0.0106	0.1503b	−0.0433	0.9821e	−0.00106	0.1507b
5	20	−0.0112	0.1582b	−0.0112	0.1582b	−0.0433	0.9821e	−0.0112	0.1582b
	100	−0.0138	0.7911b	−0.0138	0.7911b	−0.0433	0.9821e	−0.0138	0.7911b
	110	−0.0138	0.8702b	−0.0138	0.8702b	−0.0433	0.9821e	−0.0138	0.8702b
	125	−0.2153	0.3008n			−0.2153	0.3008n	−0.2258	0.3001t
		−0.0138	0.9493b	−0.0138	0.9493b	−0.0433	0.9821e	−0.0138	0.9493b
	125	−0.1956	0.5035n			−0.1956	0.5035n	−0.2264	0.5021t
		−0.0433	0.9821e			−0.0433	0.9821e		
		−0.0138	0.9888b	−0.0138	0.9888b			−0.0138	0.9888b

In Table 2 we show some dimensionless characteristics of the natural values of η/m and the characteristics of the attenuation factors of the corresponding forms $k\chi_0$/m for orthotropic cylindrical perspex shells of open profile with mechanical parameters (4.5) and the following geometrical parameters: $R=40$, $r_0=0.0637$, $h=1/50$, $k=0.7851$ and $b=4$ (b is the distance between the boundary generatrices), and l = 15.5. The moduli of elasticity E_1 and E_2 correspond to directions along the generatrices and directrices respectively.

Table 3

		$\eta_1=\eta_2=\eta_3=\eta$		$\eta_1=\eta_2=0,\ \eta_3=\eta$		$\eta_1=\eta_2=\eta,\ \eta_3=0$		$\eta_2=\eta_3=\eta,\ \eta_1=0$	
		$k\chi_0/m$	η/m	$k\chi_0/m$	η/m	$k\chi_0/m$	η/m	$k\chi_0/m$	η/m
	2					−0.0973	0.9559e		
	3					−0.0612	0.9785e		
	4	−0.0073	0.0403b	−0.0073	0.0403b	−0.0569	0.9811e	−0.0073	0.0403b
	5	−0.0136	0.0504b	−0.0136	0.0504b	−0.0558	0.9817e	−0.0136	0.0504b
	10	−0.0174	0.1008b	−0.0174	0.1008b	−0.0551	0.9821e	−0.0174	0.1008b
	11	−0.0175	0.1108b	−0.0175	0.1108b	−0.0551	0.9821e	−0.0175	0.1108b
	12	−0.0176	0.1209b	−0.0176	0.1209b	−0.0551	0.9821e	−0.0176	0.1209b
15	13	−0.0176	0.1310b	−0.0176	0.1310b	−0.0551	0.9821e	−0.0176	1.1310b
	20	−0.0176	0.2015b	−0.0176	0.2015b	−0.0551	0.9821e	−0.0176	0.2015b
	90	−0.0176	0.8565b	−0.0176	0.9068b	−0.0551	0.9821e	−0.0176	0.9068b
	95	−0.2692	0.3533n			−0.2692	0.3533n	−0.2878	0.3524t
		−0.0176	0.9572b	−0.0176	0.9572b			−0.0176	0.9572b
		−0.0551	0.9821e			−0.0551	0.9821e		
	100	−0.2373	0.5694n			−0.2373	0.5694n	−0.2888	0.5678t
		−0.0551	0.9821e	−0.0176	1.0076b	−0.0551	0.9821e	−0.0176	1.0076b
	2					−0.1170	0.9355e		
	11					−0.0554	0.9819e		
	12	−0.0058	0.1209b	−0.0074	0.1209b	−0.0553	0.9820e	−0.0074	0.1209b
	13	−0.0098	0.1310b	−0.0105	0.1310b	−00552	0.9821e	−0.0105	0.1310b
	14	−0.0119	0.1411b	−0.0119	0.1411b	−0.0551	0.9821e	−0.0119	0.1411b
	15	−0.0133	0.1511b	−0.0133	0.1511b	−0.0551	0.9821e	−0.0133	0.1512b
	16	−0.0143	0.1612b	−0.0145	0.1612b	−0.0551	0.9821e	−0.0145	0.1612b
5	17	−0.0150	0.1713b	−0.0152	0.1713b	−0.0551	0.9821e	−0.0152	0.1713b
	20	−0.0163	0.2015b	−0.0163	0.2015b	−0.0551	0.9821e	−0.0163	0.2015b
	90	−0.0176	0.9068b	−0.0176	0.9068b	−0.0551	0.9821e	−0.0176	0.9068b
	95	−0.2692	0.3533n			−0.2692	0.3533n	−0.2878	0.3524t
		−0.0176	0.9572b	−0.0176	0.9572b			−0.0176	0.9572b
		−0.0551	0.9821e			−0.0551	0.9821e		
	100	−0.2373	0.5694n			−0.2373	0.5694n	−0.2888	0.5678t
		−0.0551	0.9821e	−0.0176	1.0076b	−0.0551	0.9821e	−0.0176	1.0076b

The results shown in Table 3 correspond to closed cylindrical perspex shells with mechanical parameters (4.5) and the following geometrical parameters: $R = 40$, $r_0 = 1/20$, $h = 1/50$, $k = 1$, and l = 15.5. The following quantities were taken as the characteristic of the attenuation factors

$$k\chi_0/m = \max\{k\mathrm{Re}\chi_1/m, k\mathrm{Re}\chi_2/m, k\mathrm{Re}\chi_3/m, k\mathrm{Re}\chi_4/m\} \quad (4.6)$$

In Tables 2 and 3, after the characteristics of the attenuation factors and the natural frequencies we indicate the type of vibrations: b is predominantly flexural, e is predominantly plane, n is a new type of vibration, and t is predominantly torsional.

In Tables 2 and 3 the case $\eta_1 = \eta_2 = \eta_3 = \eta$ corresponds to problems (1.1)–(1.3), (1.5) and (1.1)–(1.4). The case $\eta_1 = \eta_2 = 0$ and $\eta_3 = \eta$ corresponds to problems (1.1)–(1.3), (1.5) and (1.1)–(1.4), in which there are no shear components of the inertial force, i.e. we have predominantly a flexural type of vibrations. The case $\eta_1 = \eta_2 = \eta$ and $\eta_3 = 0$ corresponds to a predominantly plane type of vibrations, while $\eta_2 = \eta_3 = \eta$ and $\eta_1 = 0$ corresponds to a predominantly flexural-torsional type of vibrations. The first frequencies of natural vibrations, localized at the free edge of cylindrical shells, where there is a normal component of the inertial force, are the vibration frequencies of a predominantly flexural type of vibrations.

Calculations show that, together with the first frequencies of a quasi-transverse type of vibrations, there are also frequencies of non-decaying vibrations of the quasi-shear type. These vibrations become Rayleigh-type vibrations as m increases.

When $\varepsilon_m \to 0$, the natural vibrations for (1.1)–(1.4) and (1.1)–(1.3) and (1.5) are split into quasi-transverse and quasi-shear vibrations. The frequencies of these problems approach the frequencies of the similar problems for a plate-strip and a rectangular plate respectively. The quasi-transverse type of vibrations become non-decaying vibrations as m increases. The dimensionless characteristics η_m of the natural frequency of the quasi-shear vibrations approaches the root of the Rayleigh equation $K_2(\eta_m) = 0$ (for perspex $\eta_m^{(2)} \approx 0.9821$).

When ml$\to \infty$ the natural frequencies of problem (1.1)–(1.4) approach the natural frequencies of vibrations localized at the free end of a semi-infinite closed cylindrical shell, while the natural frequencies of problem (1.1)–(1.3) and (1.5) approach the natural frequencies of vibrations localized at the free edge of a semi-infinite cylindrical shell of open profile when there is a Navier hinge mounting on the boundary generatrices.

Depending on the parameter a^2m^2, no more than two new types of vibrations occur, with characteristics solely for cylindrical shells, due to the longitudinal and torsional components of the inertia.[6,10,16] For predominantly shear type vibrations of cylindrical shells ($\eta_1 = \eta_2 = \eta$, $\eta_3 = 0$) in addition to "Rayleigh" type plane vibrations, there may also be no more than two new vibrations, also due to longitudinal and torsional components of the inertia.[6,10,16] When there is no normal component of the inertia, shear localized vibrations appear at lower wave numbers m. Vibrations of the predominantly flexural-torsional type ($\eta_2 = \eta_3 = \eta$, $\eta_1 = 0$) for fairly large m can be split into quasi-transverse and predominantly torsional vibrations. When m increases further, the quasi-transverse type vibrations become non-decaying vibrations (the parameters of the non-decaying vibrations are not given in the table; the gaps in the table correspond to positions where no frequencies of decaying vibrations are detected.

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